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Longitudinal Plasma Oscillations

HAROLD WEITZNER

1 June 1965

AEC Research and Development Report

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Magneto-Fluid Dynamics Division
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C. 1

Abstract

The problem of the plane wave oscillation of a one-dimensional plasma is considered. Various asymptotic expansions of the solution for a stable plasma are presented. First, an expansion is given for short time valid for arbitrary wavelengths. Second, expansions are given for long wavelengths valid over a long time interval. The time interval is sufficiently long to observe exponential damping of the plasma oscillations. Whenever the oscillations are strongly damped there remains an incoherent part of the electric field that slowly and irregularly tends to zero. Exponential damping is observable, however, only in the long wavelength limit.

The mixed initial value-boundary value problem is solved for the case of harmonic time-dependent data given on a wall. It is shown that for stable plasmas the electric field approaches a limiting form in time. The specular reflection boundary condition is solved as a special case. The limiting electric field is characterized by an analyticity condition analogous to those obtained from causality conditions in scattering theory. Finally, asymptotic expansions are given with results analogous to those described above where the small parameter analogous to wave number is the deviation of the frequency of oscillation from the plasma frequency.

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I. Introduction

One of the simplest problems in plasma physics is that of the linearized theory of the plane wave oscillations of a plasma composed of electrons and a background of fixed ions[1]. The system is described by an unperturbed spatially homogeneous distribution function of electrons $\tilde{f}_o(u)$, the perturbed electron distribution function $f(x,u,t)$ and the electric field $E(x,t)$. The distribution function and the electric field are determined by the linearized Vlasov equation

$$(1) \qquad \frac{\partial f(x,u,t)}{\partial t} + u \frac{\partial f}{\partial x} + \frac{q}{m} E(x,t) \frac{\partial \tilde{f}_o(u)}{\partial u} = 0$$

and Poisson's equation

$$(2) \qquad \frac{\partial E}{\partial x} = 4\pi q \int f(x,u,t) du.$$

The initial value problem for a plane wave solution was first solved by L. Landau in 1945[2], and there has been extensive discussion of the interpretation of the solution ever since. In the same paper the boundary value problem for the system with harmonic time dependence of $\tilde{f}(x,u,t)$ and $E(x,t)$ was also solved for a special $f_o(u)$ and with specular reflection at a wall. In this paper we aim to clarify some questions concerning both problems raised by us[3] and by others.

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[1] The literature on this problem is enormous and we mention only three selected references. The earliest mention of plasma oscillations is in a paper of L. Tonks and I. Langmuir, Phys. Rev. 33, 195 (1929). L. Landau, J. Phys. (U.S.S.R.) 10, 25 (1945) gave the first satisfactory account starting from the linearized Vlasov equation. G. Backus, J. Math. Phys. 1, 178 (1960) has examined this problem with some care for mathematical problems.

[2] See 1.

In his discussion of the initial value problem where $f(x,u,t)$ and $E(x,t)$ have spatial dependence e^{ikx}, Landau considered the case in which $\tilde{f}_o(u)$ is a Maxwell distribution. Under the assumption that $f(x,u,o)$ is analytic in an appropriate strip in velocity space, he obtained an asymptotic expansion of the electric field for large time. This expansion indicated that the electric field oscillated and damped in time. For small k (long wavelength) the damping was very small and the oscillation frequency was near the characteristic plasma frequency. For large k (short wavelength) the damping was very strong and the oscillation frequency was far from the plasma frequency. There were many later "physical" derivations of the oscillation and especially the damping -Landau damping- which obtained analogous results for cases other than Maxwell distributions and analytic initial data. However, to our knowledge none of these derivations is valid nor does any give a full description of the behavior of the electric field. We emphasized in an earlier paper (I) that it is not physically reasonable to consider only analytic functions for initial data and that without this assumption the electric field does not have an asymptotic expansion of the form described by Landau. We showed later (II), however, that plasma oscillations were present in the long wavelength limit. We were unable to find a complete description of the dispersion of the plasma oscillations in time and we said that Landau damping was not observable.

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[3] We refer to three of our earlier papers on this problem: I. H. Weitzner, Phys. Fluids 6, 1123 (1963), II. H. Weitzner, Phys. Fluids 7, 476 (1964), III. H. Weitzner, Comm. Pure Appl. Math. 18, 307 (1965). We refer to these papers as I, II, and III.

We show below that in the long wavelength limit, but only in this case, it is possible to obtain plasma oscillations and observable Landau damping in a limited time interval. Asymptotically in time no oscillations are generally present and there is a slow irregular dispersion of the electric field to zero. If the wavelength is not long then our previous results apply, and after an initial coherent form the electric field disperses to zero. The explicit nature of the dispersion of the electric field depends extremely sensitively on the detailed properties of the initial data, and such information is not physically meaningful. The general behavior of the solution is of interest in fields other than plasma physics since it indicates how a non-dissipative, "collisionless" system may, in appropriate circumstances, approximate the damping normally associated with "collisional" effects. One certainly expects comparable phenomena in other problems of kinetic theory and statistical mechanics.

For the problem in question we may rewrite the system (1), (2) as

$$\frac{\partial f}{\partial t} + \mathcal{O} f = 0,$$

where $\mathcal{O}$ is a linear operator. For stable plasmas (we describe the concept more precisely below) $\mathcal{O}$ has only a continuous spectrum, the whole imaginary axis. But in the long wavelength case there is an approximate point eigenvalue on or near the imaginary axis which induces plasma oscillations. Thus, in this limit, the problem is one of the separation of the contributions to the electric field from the neighborhood of the approximate point eigenvalue and the remainder of the spectrum and then the estimation of their behavior in time. We are now able to extend our earlier results by considering double expansions of the electric field in terms of k and t in order to find a more complete history of the plasma oscillations. For a short interval we give an expansion of the electric field that treats all parts of the continuous spectrum on an equal basis. If we consider small values of k in this expansion, then we may identify terms corresponding to plasma oscillations and contributions from the remainder of the continuous spectrum. For the remainder of the expansions we take k small and t as specified. For t of the order of one, or not too large, we still may follow the plasma oscillations, but we may only estimate the contribution of the remainder of the continuous spectrum. In some cases (but not for Maxwell distributions or other with infinitely many moments), approximate exponential damping in time may be observable in this time interval. As time increases, but again is not too large, we find sharper estimates on the contribution of most of the continuous spectrum and we are able to observe exponential damping of the plasma oscillations. Finally, for sufficiently large time the coherent plasma oscillation has dispersed and there remains only a slow irregular approach to equilibrium. The details of these expansions are given in the second section.

The boundary value problem studied by Landau is rather confused in that even for the special case considered there are no indications of proofs or ranges of validity for the expansions that exhibit plasma oscillations. Rather than consider only the boundary value problem, we treat the mixed initial value-boundary value problem and then approach the limit of large time. We take the prescribed boundary values to have the time dependence $e^{-i\omega t}$. In the special case of specular reflection we obtain the well-known result. We choose this circuitous procedure to select the one causal solution of the infinitely many solutions of the system (1), (2) with time dependence $e^{-i\omega t}$. We show, as should be expected, that the solution is uniquely characterized as the one which possesses an analytic continuation in the full upper half-plane of the complex variable ω. The approach to equilibrium applies only to the case of stable plasmas. Finally we consider asymptotic expansions of the solution corresponding to those given for the initial value problem. We obtain comparable results except that the expansion for small x no longer exhibits plasma oscillations as did the one previously obtained for small t. We give the solution to the initial value-boundary value problem and discuss the large time limit in Section III. In Section IV we examine the expansions that show the plasma oscillations.

II. The Initial Value Problem

If we assume that $f(x,u,t)$ and $E(x,t)$ have spatial dependence e^{ikx}, so that $f(x,u,o)=\tilde{g}(u)e^{ikx}$, then we may solve the system (1), (2) easily by taking a Laplace transform in time and eliminating $f(x,u,p)$ in favor of $E(x,p)$. After some changes of variables we find

$$(3) \qquad E(x,t)=2qk\lambda^2 e^{ikx}\int_{-\infty+i\sigma}^{\infty+i\sigma}\frac{dz G(z)e^{-ikvtz}}{(k\lambda)^2-F(z)} \,,$$

where $\omega_p = (\frac{4\pi nq^2}{m})^{\frac{1}{2}}$ is the plasma frequency, v is the thermal speed used to non-dimensionalize the distribution functions so that

$$\tilde{f}_o(u) = \frac{n}{v} f_o(\frac{u}{v})$$

$$\tilde{g}(u) = \frac{1}{v} g(\frac{u}{v}),$$

n is the electron density (chosen so that $\int_{-\infty}^{\infty} f_o(y)dy = 1$), $\lambda = \frac{v}{\omega_p}$ is the Debye distance and

$$(4) \qquad G(z) = -\int_{-\infty}^{\infty} \frac{g(y)dy}{y-z}, \qquad Imz \geq 0$$

$$(5) \qquad F(z) = \int_{-\infty}^{\infty} (\frac{\partial f_o(y)}{\partial y}) \frac{dy}{y-z}, \quad Imz \geq 0.$$

The contour of integration in (3) is chosen so that the integrand is analytic in $Imz \geq \sigma$. We show below that $F(z) = \mathcal{O}(1/z^2)$, and hence such a σ always exists. We note that the original equations (1), (2) are invariant under a Gallilean transformation $x' = x-at$, $t' = t$, $u' = u-a$. Thus without loss of generality we may choose the reference frame in which $\int_{-\infty}^{\infty} yf_o(y)dy = 0$.

If F(z) is real and positive for some z in the upper half-plane, then clearly E(x,t) grows exponentially in time and we call the plasma unstable. For our purposes it suffices to describe a plasma as stable if F(z) is not real and positive in $\text{Im} z \geq 0$. We have included the real axis in the definition of stability and, in fact, excluded cases in which the electric field is bounded or even tends to zero, but we avoid many pathological cases. In the next section we admit unstable plasmas in considering the mixed initial value-boundary value problem before discussing the approach to equilibrium for stable plasmas. For the remainder of this section we consider only stable cases. The Maxwell distribution function and any other positive distribution function with a single maximum which tends to zero sufficiently rapidly at infinity are examples of stable systems, although many others exist.[4]

In order to proceed we need various asymptotic expansions of $F(z)$ and $G(z)$ for large $|z|$ in $\text{Im} z \geq 0$. We omit the details which are given in I and III and which are quite straightforward. If $f_o(y)$ has $N+4$ moments then we readily obtain

$$(6) \qquad F(z) = \frac{1}{z^2}\left(1 + \sum_{n=2}^{N} \frac{\beta_n}{z^n} + O\left(\frac{1}{z^{n+1}}\right)\right) ,$$

where β_n is just a multiple of the n^{th} moment of $f_o(y)$ and the expansion is valid in $\text{Im} z \geq 0$. If $f_o(y)$ has infinitely many moments then we take any number of terms in the series that we choose, provided $|z|$ is sufficiently large. For real z the expansion of $F(z)$ has no imaginary part. The presence of an imaginary part is essential to move the approximate zero of the denominator of (3) in the z plane from the real axis to the lower half-plane and thus induce approximate exponential damping of the electric field. We refer to distribution functions with infinitely many moments as type I distribution functions. We also examine a second type of distribution function with only a finite number of moments and with the asymptotic expansion

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[4] O. Penrose, Phys. Fluids 3, 258 (1960) has given a simple characterization of stable plasmas.

$$\frac{\partial f_o(y)}{\partial y} \sim - \operatorname{sgn}(y) \sum_p \frac{\gamma_p}{|y|^p},$$

in which case $F(z)$ has an asymptotic expansion of the form

$$(7) \qquad F(z) = \frac{1}{z^2}\left(1 + \sum_{n=2}^{N} \frac{\beta_n}{z^n} + O\left(\frac{1}{z^{N+1}}\right)\right) - \pi \sum_p \frac{\gamma_p e^{i\pi p}}{\sin \pi p}\left(\frac{1}{z^p} + \frac{1}{(-z)^p}\right) ,$$

(if p is an integer then the term in the second summation is replaced by $-\frac{1}{\pi}\left\{(\log z - i\pi)/z^p + (\log z)/(-z)^p\right\}$). We note the presence of an imaginary part in these series for real z. Of course $G(z)$ has similar expansions which for convenience we take of the form

$$(8) \qquad G(z) = \sum_{n=1}^{N} \frac{\alpha_n}{z^n} + O\left(\frac{1}{z^{N+1}}\right) .$$

Finally, provided $f_o(y)$ and $g(y)$ have many well-behaved derivatives, $F(z)$ and $G(z)$ have derivatives on the real axis and we may use Taylor series expansions about any point on the real axis and the expansions are in power of $(z-z_o)$. Thus, up to specified errors we may replace $F(z)$ and $G(z)$, which are analytic only in the upper half-plane, by approximations which possess continuations into the lower half-plane, and then attempt to evaluate the integrals.

We now proceed to the simplest of the expansions. We take kvt small and do not specify $k\lambda$. We choose $\sigma = (kvt)^{-1}$, so that z is uniformly large on the contour of integration and we may expand $F(z)$ and $G(z)$ in asymptotic series. We write the solution for the explicit case of three terms in the expansion of F and G and the general error estimate. We find

$$(9) \quad E(x,t) = 4\pi q k\lambda^2 e^{ikx} \left\{ \frac{(\alpha_1 z_p^2+\alpha_3)\cos(kvtz_p) - i\alpha_2 z_p \sin(kvtz_p)}{\sqrt{1+4\beta_2(k\lambda)^2}} + \frac{(\alpha_1 z_c^2-\alpha_3)\cosh(kvtz_c) - i\alpha_2 z_c \sinh(kvtz_c)}{\sqrt{1+4\beta_2(k\lambda)^2}} + O\left(\frac{(kvt)^{N+1}}{(k\lambda)^4}\left(1 + \left(\frac{kvt}{k\lambda}\right)^2\right)\right)\right\},$$

where

$$\text{(10a)} \qquad z_p^2 = \frac{1}{2(k\lambda)^2}\left(1 + \sqrt{1+4\beta_2(k\lambda)^2}\right)$$

$$\text{(10b)} \qquad z_c^2 = \frac{1}{2(k\lambda)^2}\left(\sqrt{1+4\beta_2(k\lambda)^2} - 1\right).$$

To obtain the above error estimate we take the asymptotic expansions of $F(z)$ and $G(z)$ up to and including the terms $\mathcal{O}(1/z^N)$, replace the denominator in the error term by its minimum and, integrate the rest of the error term. Thus, for any $k\lambda$, if kvt is sufficiently small and if we take enough terms in the series, (9) gives an explicit expansion of the electric field.

Neither the original expression for the electric field, (3), nor the expansion is regular for $(k\lambda)$ small, and we examine this limit with much greater care below. For $(k\lambda)$ of order one or larger we believe that the only general properties available are (9) and the result obtained by setting $\sigma = 0$ in (3) and repeated integration by parts:

$$\text{(11)} \quad E(x,t) = 4\pi q k\lambda^2 e^{ikx} \; \mathcal{O}\left\{\frac{1}{(kvt)^n \min\limits_{z \text{ real}} |(k\lambda)^2 - F(z)|^{n+1}}\right\}.$$

For $k\lambda$ of order one or larger we have the expansion for (kvt) small and the estimate (11) for kvt large. In these cases (9) does not exhibit oscillations observable over several periods and hence does not represent a true oscillation. The estimate (11) for kvt large exhibits the dispersion of the coherent form of the electric field (9).

In the remainder of this section we take $(k\lambda)$ small. In this case there are two roots of the asymptotic expansion of $(k\lambda)^2 - F(z)$ that are $O(1/k\lambda)$ and all the other roots are $O(1)$ in $k\lambda$. Clearly $\pm z_p$ above represent the roots $O(1/k\lambda)$ and $\pm z_c$ are the roots $O(1)$ in $k\lambda$. We note that $kvtz_p \cong \omega(k)t$, where

$$\omega^2(k) = \omega_p^2 + \beta_2(kv)^2 \tag{12}$$

so that the first term in (9) corresponds to plasma oscillations. Further, $\pm z_p$ correspond to the approximate zeros of $(k\lambda)^2-F(z)$, or the approximate point eigenvalues of the system. Also $\pm z_c$ correspond to the lowest order contributions of the part of the continuous spectrum away from the approximate point eigenvalues. We denote by $E_p(x,t)$ those terms in (9) that correspond to the roots $\pm z_p$.

If we consider type I distribution functions then the expansion of $(k\lambda)^2 - F(z)$ to any order is a polynomial in $1/z$ with real coefficients and hence z_p can never aquire an imaginary part. If we consider type II distribution functions, then the expansion to a sufficiently high order will have an imaginary part so that $E_p(x,t)$ is multiplied by $e^{-kvt\delta}$, where to lowest order in $k\lambda$

$$(13) \qquad \delta = -\frac{\pi}{2}\frac{1}{(k\lambda)^3}\frac{\partial f_o}{\partial u}\bigg|_{u = (k\lambda)^{-1}} \ .$$

The above expression is the usual form of the damping present, and our primary task in this section is to show that for $(k\lambda)$ small there is a time interval in which such damping is generally observable. Surely for kvt and $k\lambda$ both small no such damping is significant.

We now proceed to the expansion of (3) based on the assumption that $k\lambda$ is small. Initially we take kvt no more than of order one or somewhat larger. We deform the contour of integration to that shown in Fig. 1, where we also define points $\pm Z_1$. We treat separately the integrals from $-Z_1$ to $+Z_1$ and the integrals from $\pm Z_1$ to infinity. In principle we should either use mollifiers or carry the end point terms in order to prove that the end point terms do not contribute to the results below. Either technique easily shows that no end point terms occur and we omit the details. Since $(k\lambda)$ is small we may take Z_1 large and use the asymptotic expansions for $|z| \geq Z_1$. In the integral from $-Z_1$ to $+Z_1$ we integrate by parts n times, replace the denominator by its minimum and estimate the resulting integral. On the other sections of the contour we use the expansions of F(z) and G(z), picking up an error term, deform the contour into the lower half-plane, picking up a contribution from the poles at $\pm z_p$, and we then integrate by parts n times. We find easily that

$$(14)\qquad E(x,t) = E_p(x,t) + \frac{e^{(kvt\sigma)}}{\left((k\lambda)^2\sigma\right)^2}\, O\!\left(\frac{1}{z_1^{N+1}}\right) kq\lambda^2$$

$$+ (kq\lambda^2)\, O\left\{\frac{\log Z_1}{\left((k\lambda)^2 - \frac{1}{z_1^2}\right)^{n+1} (kvtz_1)^n}\right\},$$

where $E_p(x,t)$ corresponds to the oscillating part of (9) and has the damping factor $\exp(-\delta\ kvt)$ for type II functions.

We may use (14) to consider many special cases by appropriate choices of σ and Z_1. For kvt of order one or smaller we take $\sigma = 1$, $Z_1 = (k\lambda)^{-r}$, $1 > r > 0$ and we readily verify that $E_p(x,t)$ is the largest term in (14). While the second error term, which comes from the continuous spectrum not near z_p, is absolutely large it is small relative to $E_p(x,t)$. At a later time this term is absolutely small as well. We note that we no longer have an explicit approximation for the contribution of the major part of the continuous spectrum, and we only have the estimate given. We might use (14) to follow the transition from kvt of order one to kvt large, but we omit the intermediate values and take kvt large but not too large. We choose $\sigma = (kvt)^{-1}$ and the previous value for Z_1. If (kvt) is in the interval

$$(15)\qquad (k\lambda)^{-N_1} \gg (kvt) \gg (k\lambda)^{-N_2},$$

then the error estimate is absolutely small and the plasma oscillation contribution is dominant, provided it has not damped too much for type II distribution functions. Again type I functions exhibit no damping. For type II functions damping is observable since $\partial f_o/\partial v \sim 1/|v|^p$ and $kvt\delta$ may be large and yet (15) may hold. Even in this time interval the amplitude of the plasma oscillations may become so small that the error terms dominate since $E_p(x,t)$ damps exponentially fast and the error terms only damp algebraically.

We now proceed to consider the limit of (kvt) very large and find damping generally. For type II functions this expansions and the preceding one may overlap. We move the entire contour of integration onto the real axis and we define intervals of length 2η centered on $\zeta_\pm$, where $\mathrm{Re}\, F(\zeta_\pm) = (k\lambda)^2$, $\zeta_+ > 0 > \zeta_-$, and $\zeta_\pm$ are large. We treat these intervals and the remainder of the real axis separately. From the parts of the integral away from $\zeta_\pm$ we find after n integrations by parts (again we drop end point terms which cancel out) that the contribution to the electric field is

$$E_1 = qk\lambda^2\, O\left\{\frac{1}{(kvt)^n|i\varepsilon_\pm + \eta(k\lambda^3)|^{n+1}}\right\} ,$$

where $\varepsilon_\pm = \mathrm{Im}\, F(\zeta_\pm)$, and we have used the Taylor series for $F(z)$ near $\zeta_\pm$ to estimate the smallest value of the denominator. We may approximate the remainder of the integral as

$$E_{\pm} = 2qk\lambda^2 \times$$

$$\int_{\zeta_{\pm}-\eta}^{\zeta_{\pm}+\eta} \frac{e^{-i(kvt)z}dz\left\{G(\zeta_{\pm})+G'(\zeta_{\pm})(z-\zeta_{\pm})+\cdots+\frac{G^{(N)}(\zeta_{\pm})}{N!}(z-\zeta_{\pm})^N\right\}}{-i\varepsilon_{\pm} - F'(\zeta_{\pm})(z-\zeta_{\pm}) + \cdots - \frac{F^{(N)}(\zeta_{\pm})}{N!}(z-\zeta_{\pm})^N}$$

$$+ kq\lambda^2 \,\mathcal{O}\left\{\frac{(k\lambda\eta)^{N+1}(k\lambda)^2}{|-i\varepsilon+(k\lambda)^3\eta|^2}\right\} .$$

We now deform the path of integration into the lower half-plane and pick up the contributions from the poles near $\zeta_{\pm}$. We note that there is no difficulty with type I distribution functions, but for type II distribution functions we may not take as many terms in the Taylor series as we please. For in the latter case $\varepsilon_{\pm}/F^{(n)}(\zeta_{\pm})$ will be large for n sufficiently large, corresponding to a large shift in the zero of the denominator from $\zeta_{\pm}$. Hence in this case we must restrict n. Finally we integrate by parts many times and duplicate the previous error estimate to find

$$(16) \qquad E(x,t) = E_p(x,t) + \mathcal{O}\left\{\frac{1}{(kvt)^n(\varepsilon^2+\eta^2(k\lambda)^6)^{\frac{n+1}{2}}}\right\} kq\lambda^2$$

$$+ (kq\lambda^2)\,\mathcal{O}\,\frac{(k\lambda\eta)^{N+1}(k\lambda)^2}{(\varepsilon^2+\eta^2(k\lambda)^6)} \quad ,$$

where we now include the exponential damping with decrement δ of (13). If we take $kvt(k\lambda)^3\eta$ large and $\eta(k\lambda)^3$ small but much larger than ε (as we may easily do), then the error terms are small and the net damping $kvt\delta = (kvt)\varepsilon(k\lambda)^{-3}$ may be $O(1)$ and hence observable. Of course when $(kvt\delta)$ becomes very large the plasma oscillations have damped out and all that is left is the slow dispersion of the electric field to zero for which a best estimate is $O\left\{1/(kvt\varepsilon)^n\right\}$. Thus in a limited time interval and when $k\lambda$ is small, exponential damping of plasma oscillations is observable.

III. The Mixed Initial Value-Boundary Value Problem

In this section we consider the original set of equations in the half-space $x \geq 0$. At the wall $(x=0)$ we prescribe the electric field and the distribution functions of particles which leave the wall. To complete the boundary conditions we require that the disturbances all be bounded for x sufficiently large for any fixed t. The initial conditions is just that imposed for the infinite problem; we give the distribution function of particles for all x. These conditions are reasonable physically, and we see below that they lead to a well-posed mathematical problem We carry out the solution explicitly only for the case of harmonic time dependence of the data given on the wall, although the methods obviously apply more generally. We show that for a stable plasma the electric field has a limit as t tends to infinity and that the limiting electric field is uniquely characterized by the prescribed boundary data and an analyticity condition analogous to that obtained from causality conditions in scattering theory. At the end of this section we obtain the solution for a specular reflection boundary condition as a special case.

We take the initial data and conditions on the boundary as

$$(17)\qquad \begin{cases} f(x,v,0) = g(x,v) \\ f(0,v,t) = \begin{cases} h^{+}(v)e^{-i\omega t} & , v > 0 \\ h^{-}(v,t) & ,v < 0 \end{cases} \\ E(0,t) = E_{o}\ e^{-i\omega t} \end{cases}$$

where all the functions are given except $h^{-}(v,t)$, which is determined by the condition that $E(x,t)$ and $f(x,v,t)$ be bounded as x tends to infinity at any fixed time. We perform Laplace transforms of the system (1), (2), both in time and space and let p represent the transform variable corresponding to t and s represent the transform corresponding to x. We expect the double transforms $E(s,p)$ and $f(s,v,p)$ to exist and further for Re p sufficiently large we expect $E(s,p)$ and $f(s,v,p)$ to be analytic in Re $s > 0$ and have at most simple poles on Re $s = 0$. We do not consider the analyticity condition on f(s,v,p) further below, for an elementary computation shows that if E(s,p) is analytic in Re $s > 0$ then f(s,v,p) is as well. The transforms of the system (1), (2) are

$$(p+sv)f(s,v,p) + \frac{qn}{m} E(s,p) \frac{\partial f_o(o)}{\partial v} = g(s,v) + \begin{cases} \dfrac{vh^+(v)}{p+i\omega}, & v > 0 \\ vh^-(v,p), & v < 0 \end{cases}$$

$$sE(s,p) - 4\pi q \int_{-\infty}^{\infty} f(s,v,p)dv = \frac{E_o}{p+i\omega} .$$

On elimination of f(s,v,p) from the equations and with the definitions

$$F(z) = \int_{-\infty}^{\infty} \left(\frac{dv}{v+z}\right) \frac{\partial f_o(v)}{\partial v} \tag{18}$$

$$G(z,s) = 4\pi q \int_{-\infty}^{\infty} \left(\frac{dv}{v+z}\right) g(s,v) \tag{19}$$

$$H(z) = 4\pi q \int_{0}^{\infty} \left(\frac{dv}{v+z}\right) vh^+(v) \tag{20}$$

$$K(z,p) = 4\pi q \int_{-\infty}^{0} \left(\frac{dv}{v+z}\right) vh^-(v,p) \tag{21}$$

we find

$$(22) \qquad E(s,p) = \frac{\frac{sE_o+H(p/s)}{p+i\omega} + K(p/s,\ p) + G(p/s,\ s)}{s^2 + \omega_p^2\ F(p/s)}\ .$$

Clearly $F(z)$ has an expansion of the form (7).

We see from (18) - (21) that $F(z)$ and $G(z)$ are defined for all values in the complex z plane except for real z and that $F(z)$ and $G(z)$ have cuts on the real axis. Further $H(z)$ $(K(z,p))$ has a cut on the negative (positive) real axis, and both functions are bounded at the origin and tend to zero at infinity. As (22) stands, $E(s,p)$ is discontinuous across $p/s = z$ real and positive, and $E(s,p)$ may be singular at any of the zeros of the denominator of (22). We must choose $h^-(v,p)$ so that in Re $p > k$, Re $s > 0$, $E(s,p)$ is analytic in both variables. We may write the denominator of (22) as $\frac{s^2}{p^2}\ (p^2 + \omega_p^2\ (p/s)^2F(p/s))$. Since $|z^2F(z)|$ is bounded by some constant it follows that for Rep sufficiently large $E(s,p)$ has no poles. We ultimately extend our expression for $E(s,p)$ to all p in Re $p \geq 0$ for stable plasmas. In order to suppress the cut on $z = p/s$ real and positive it suffices to require that $E(\frac{p}{z},p)$ be continuous across z real and positive. We may restate this condition as a Hilbert problem for $K(z,p)$:

$$(23) \qquad K^{+}(z,p) = \left\{ \frac{p^2+\omega_p^2 z^2 F^{+}(z)}{p^2+\omega_p^2 z^2 F^{-}(z)} \right\} K^{-}(z,p)$$

$$+ \left(\frac{E_o \; p/z + H(z)}{p+i\omega} \right) \left\{ \left(\frac{p^2+\omega_p^2 z^2 F^{+}(z)}{p^2+\omega_p^2 z^2 F^{-}(z)} - 1 \right) \right\}$$

$$+ G^{-}(z, \tfrac{p}{z}) \left(\frac{p^2+\omega_p^2 z^2 F^{+}(z)}{p^2+\omega_p^2 z^2 F^{-}(z)} \right) - G^{+}(z, \tfrac{p}{z}),$$

z real and positive,

where the ± refers to the limit taken from above or below the real axis.

To solve (23) we start by defining

$$(24) \qquad \Gamma(z) = \frac{1}{2\pi i} \int_{o}^{\infty} \left(\frac{dt}{t-z}\right) \log \left(\frac{p^2+\omega_p^2 t^2 F^{+}(t)}{p^2+\omega_p^2 t^2 F^{-}(t)} \right),$$

and in (24) we select the branch of the logarithm that tends to zero as t tends to infinity. Again for Rep sufficiently large the logarithm tends to zero as t tends to zero as well. Clearly $e^{\Gamma(z,p)}$ satisfies the homogeneous form of (23) and the standard arguments show that no solution of the homogeneous equation is bounded at the origin, tends to zero at infinity and is analytic except for a cut on the positive real axis. Thus the unique acceptable solution of (23) is

$$(25)\qquad K(z,p) = \frac{e^{\Gamma(z,p)}}{2\pi i}\int_0^\infty \frac{dt}{t-z}\left\{\left(\frac{(p/t)\,E_o + H(t)}{p + i\omega}\right) \times \left(e^{-\Gamma^-(t,p)} - e^{-\Gamma^+(t,p)}\right) + G^-(t,\ p/t)e^{-\Gamma^-(t,p)} - G^+(t,\ p/t)e^{-\Gamma^+(t,p)}\right\} .$$

We now wish to modify the contours of integration in (24) and (25) so as to exhibit explicitly the desired analyticity properties of E(s,p) and to obtain representations appropriate to study the approach to equilibrium.

We rewrite (24) as two integrals by the identity

$$\log \frac{p^2+\omega_p^2z^2F^+(z)}{p^2+\omega_p^2z^2F^-(z)} = \log\left(\frac{p^2+\omega_p^2z^2F^+(z)}{p^2+\omega_p^2}\right) - \log\left(\frac{p^2+\omega_p^2z^2F^-(z)}{p^2+\omega_p^2}\right) ,$$

and each integral converges separately. We take the first (second) integral and deform the contour of integration to a line perpendicular to p in the upper (lower) half plane and introduce the variable $t = ipt'$ to find

$$(26a)\qquad \Gamma(z,p) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt'}{(t'-z/ip)} \log\left(\frac{p^2(1-\omega_p^2(t')^2F(ipt')}{p^2+\omega_p^2}\right) + \log\left(\frac{p^2+\omega_p^2z^2F(z)}{p^2+\omega_p^2}\right)\cdot\eta(\mathrm{Re}\ p/z),$$

or

$$(26b) \qquad \Gamma(z,p) = \Gamma(z,p) + \log\left(\frac{p^2+\omega_p^2 z^2 F(z)}{p^2+\omega_p^2}\right)\cdot\eta(\mathrm{Re}\ \tfrac{p}{z}),$$

where $\eta(x)$ is the Heaviside step function. We perform a similar manipulation on all the terms in (25) with the exception of the term proportional to E_o for which we use the contour of integration of Fig. 2 and we obtain

$$(27) \quad K(z,p) = \frac{e^{\Gamma(z,p)}}{2\pi i}\left\{-2\pi i\ e^{-\Gamma(z,p)}\left[\frac{pE_o}{z(p+i\omega)} + \frac{H(z)}{p+i\omega} + G(z, \tfrac{p}{z})\right]\right.$$
$$+ \frac{2\pi i p E_o\ e^{-\Gamma(0,p)}}{z(p+i\omega)}$$
$$\left. - \int_0^\infty \frac{dt' e^{-\Gamma(ipt',p)}}{t' - z/ip}\left(\frac{H(ipt')}{p+i\omega} + G\left(ipt', \frac{1}{it'}\right)\right)\right\}.$$

In (27) we have taken $\mathrm{Re}(p/z)$ positive. From the definition of $E(s,p)$, the relation $p = zs$ we infer

$$(28) \quad E(s,p) = \frac{e^{\bar\Gamma(p/s,\ p)-\log(p^2+\omega_p^2)}}{2\pi i}\left\{\frac{2\pi i\ p^2 E_o\ e^{-\Gamma(0,p)}}{s(p+i\omega)}\right.$$
$$\left. - \frac{p^2}{s}\int_{-\infty}^{\infty}\left(\frac{dt' e^{-\Gamma(ipt',p)}}{(st'+i)}\right)\left(\frac{H(ipt')}{p+i\omega} + G\left(ipt', \frac{1}{it'}\right)\right)\right\}.$$

In (28) we have exhibited $E(s,p)$ as an analytic function of s and p in $\mathrm{Re}\ s > 0$, $\mathrm{Re}\ p > k$, so that (28) represents an explicit solution to the problem. Before proceeding further we note an identity derivable from (27) by taking the leading term in the asymptotic expansion for large z

$$(29)\quad \frac{1}{p+i\omega}\left(pE_0+4\pi q\int_0^\infty vh^+(v)dv\right) + 4\pi q\int_{-\infty}^{0} vh^-(v,p)dv$$

$$+\, 4\pi q\int dxdvg(x,v) = \frac{pE_0\; e^{-\Gamma(0,p)}}{p+i\omega}$$

$$+\, \frac{p}{2\pi}\int_{-\infty}^{\infty} dt'\, e^{-\Gamma(ipt',p)}\left(\frac{H(ipt')}{p+i\omega} + G\left(ipt', \frac{1}{it'}\right)\right).$$

We might use (28) and (29) to study $\lim_{x\to\infty} E(x,t)$ by obtaining the residue of (28) at $s = 0$, but the results are none too illuminating. In general this limit is not zero since Eq. (2) and the boundary conditions show that even at the initial time the electric field is already not zero at infinity. We might easily verify, however, that even for stable plasmas $\lim_{t\to\infty}\lim_{x\to\infty} E(x,t) \neq \lim_{x\to\infty}\lim_{t\to\infty} E(x,t)$.

We now restrict ourselves to stable plasmas and we examine the approach to equilibrium. We see that if $F(z)$ is never real and positive, the condition for a stable plasma, then (26) provides an analytic continuation of $\Gamma(p/s,\, p)$ into the half-plane $\mathrm{Re}\, p \geq 0$, $p = \pm\, i\omega_p$ excepted, provided $\mathrm{Re}\, s > 0$. It also follows that (26) provides an analytic continuation of $\Gamma(ipt',p)$, t' real, in the same domain in the p plane. Thus $E(s,p)$ is now well defined in Re $p \geq 0$, Re $s > 0$, $p \neq -i\omega$, $\pm i\omega_p$. Finally we note that the combinations $(\bar{\Gamma}(p/s,\ p) - \log(\omega_p^2+p^2) - \Gamma(0,p))$ and $(\bar{\Gamma}(p/s,\ p) - \log(p^2+\omega_p^2) - \Gamma(ipt',p))$ are well defined on the imaginary p axis at all points, and these combinations are differentiable with respect to p except at $p = \pm i\omega_p$. Thus $E(s,p)$ is defined in R $p \geq 0$ except at $p = -i\omega$, where it has a pole, and the standard arguments show that

$$(30)\quad \lim_{t \to \infty} (E(s,\ t)e^{i\omega t}) = -\frac{\omega^2}{s}\, e^{\bar{\Gamma}(-\frac{i\omega}{s},\ -i\omega) - \log(\omega_p^2-\omega^2)} \times \left\{E_o e^{-\Gamma(0,-i\omega)} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt'H(\omega t')e^{-\Gamma(\omega t',-i\omega)}}{(st'+i)}\right\} .$$

We study (30) in greater detail in the next section, we note that we may obtain the limiting form of (30) for large x exactly as we found the form for large t. The dominant singularity is at $s = 0$, hence

$$\lim_{x \to \infty}\ \lim_{t \to \infty} E(x,t)e^{i\omega t})$$

$$= \frac{1}{\left(1 - \frac{\omega_p^2}{\omega^2}\right)}\left\{E_o\ e^{-\Gamma(0,-i\omega)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dt'e^{-\Gamma(\omega t',-i\omega)}H(\omega t')\right\} ,$$

which by (29) becomes

$$(31)\quad \lim_{x\to\infty}\ \lim_{t\to\infty} \left(E(x,t)e^{i\omega t}\right) = \frac{E_o - \frac{4\pi q}{\omega i}\int_{-\infty}^{\infty} vh^{\pm}(v,t=\infty)dv}{1 - \frac{\omega_p^2}{\omega^2}} .$$

Thus with no current flow at the wall, e.g. for a specular reflection boundary condition, the limiting electric field is $E_o(1 - \omega_p^2/\omega^2)^{-1}$, and otherwise there is an additional contribution from the current flow at the wall. Eq. (31) is exactly what we expect for a dielectric medium with a real dielectric constant $(1 - \omega_p^2/\omega^2)$.

We conclude this section by obtaining the solution in the case of a specular reflection boundary condition. As is physically reasonable, but by no means necessary mathematically, we now assume that $f_o(v)$ is even and we look for a solution for which $h^+(v,t) = h^-(-v,t)$ (we no longer restrict $h^+(v,t)$ to the time dependence $e^{-i\omega t}$, thus $H+(z)/(p+i\omega)$ now becomes $H(z,p)$. We start from (25) perform two changes in the paths of integration. First, we treat the term proportional to E_o exactly as above. Second, we treat the two integrals containing the terms $H(t,p)e^{-\Gamma^+(t,p)}$, $H(t,p)e^{-\Gamma^-(t,p)}$ separately and use the contours of integration of Fig. 3, and after rewriting the result we find

$$(32)\quad L(z,p) = H(z,p) + K(z,p) = \frac{pE_0(e^{\Gamma(z,p)-\Gamma(0,p)} - 1)}{z(p+i\omega)}$$

$$+ \frac{e^{\Gamma(z,p)}}{2\pi i}\int_{-\infty}^{0} (\frac{dt}{t-z})e^{-\Gamma(t,p)}(H^+(t,p)-H^-(t,p))$$

$$+ \frac{e^{\Gamma(z,p)}}{2\pi i}\int_{0}^{\infty} \frac{dt}{(t-z)}\Big\{G^-(t,\ p/t)e^{-\Gamma^-(t,p)}$$

$$- G^+(t,\ p/t)e^{-\Gamma^+(t,p)}\Big\} .$$

The condition of specular reflection is just $L(z,p) = L(-z,p)$, and thus we must determine $H^+(t,p) - H^-(t,p) = -2\pi i t h^+(t,p)$ so that the right hand side of (32) is an even function of z.

We may reduce this problem to a very trivial Wiener-Hopf problem. We take the condition $L(z) = L(-z)$ and multiply by $e^{-\Gamma(z,p)-\Gamma(-z,p)}$ so that

$$(\frac{p}{z})\left(\frac{E_0}{p+i\omega}\right)\left(e^{-\Gamma(-z,p)-\Gamma(0,p)} - e^{-\Gamma(z,p)-\Gamma(-z,p)}\right)$$

$$(33)\quad + \frac{e^{-\Gamma(-z,p)}}{2\pi i}\Big\{\int_{-\infty}^{0}(\frac{dt}{t-z})e^{-\Gamma(t,p)}(H^+(t,p)-H^-(t,p))$$

$$+ \int_{0}^{\infty}(\frac{dt}{t-z})\Big(G^-(t,\ p/t)e^{-\Gamma^-(t,p)} - G^+(t,\ p/t)e^{-\Gamma^+}\Big\} =$$

$$= -\left(\frac{p}{z}\right)\left(\frac{E_o}{p+i\omega}\right)\left(e^{-\Gamma(z,p)-\Gamma(0,p)} - e^{-\Gamma(z,p)-\Gamma(-z,p)}\right)$$

$$+ \frac{e^{-\Gamma(z,p)}}{2\pi i}\left\{\int_{-\infty}^{0}\left(\frac{dt}{t+z}\right)e^{-\Gamma(t,p)}(H^+(t,p)-H^-(t,p))\right.$$

$$\left. + \int_{0}^{\infty}\left(\frac{dt}{t+z}\right)\left(G^-(t,\ p/t)e^{-\Gamma^-(t,p)} - G^+(t,\ p/t)e^{-\Gamma^+}\right)\right\} \quad .$$

We note that we may easily split (33) into the sum of two sets of terms; the first consists of functions analytic except for a cut on the positive real axis and the second contains functions analytic except for a cut on the negative real axis. Both sets of functions are bounded at the origin and go to zero at infinity; hence both sets are identically zero. We carry out the details for the case $f_o(v)$ and hence $F(z)$ even and $G(z,s) = 0$. Since the initial data do not affect the solution for large time we do not make a serious error in their omission. We use the identity

$$\Gamma(z,p) + \Gamma(-z,p) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{dt}{t-z}\right)\log\left(\frac{p^2+\omega_p^2t^2F^+(t)}{p^2+\omega_p^2t^2F^-(t)}\right),$$

or

$$(34) \qquad \Gamma(z,p) + \Gamma(-z,p) = \log(p^2+z^2\omega_p^2F(z)).$$

With these assumptions and (34), (33) becomes

$$\begin{aligned}
&\frac{E_o}{(p+i\omega)}\left(\frac{p}{z}\right)\left(e^{-\Gamma(-z,p)}-e^{-\Gamma(0,p)}\right)e^{-\Gamma(0,p)} \\
&+\frac{e^{-\Gamma(-z,p)}}{2\pi i}\int_{-\infty}^{0}\left(\frac{dt}{t-z}\right)e^{-\Gamma(t,p)}\left(H^{+}(t,p)-H^{-}(t,p)\right) \\
&-\frac{pE_o}{\pi i(p+i\omega)}\int_{-\infty}^{0}\frac{dt}{t(t-z)}\left(\frac{1}{p^2+\omega_p^2t^2F^{+}(t)}-\frac{1}{p^2+\omega_p^2t^2F^{-}(t)}\right) \\
&=-\frac{p\ E_o}{z(p+i\omega)}\left(e^{-\Gamma(z,p)}-e^{-\Gamma(0,p)}\right)e^{-\Gamma(0,p)} \\
&\quad+\frac{e^{-\Gamma(z,p)}}{2\pi i}\int_{-\infty}^{0}\left(\frac{dt}{t+z}\right)e^{-\Gamma(t,p)}\left(H^{+}(t,p)-H^{-}(t,p)\right) \\
&\quad+\frac{pE_o}{\pi i(p+i\omega)}\int_{0}^{\infty}\frac{dt}{t(t-z)}\left(\frac{1}{p^2+\omega_p^2t^2F^{+}(t)}-\frac{1}{p^2+\omega_p^2t^2F^{-}(t)}\right).
\end{aligned}\tag{35}$$

The left hand (right hand) side of (35) has a cut on the negative (positive) real axis, is bounded at the origin and tends to zero at infinity. Hence both sides of (35) are entire and vanish at infinity so that they are identically zero. After straightforward manipulations we find

$$E(s,p)=\frac{E_o}{s(p+i\omega)}\left\{1+\frac{p^3}{\pi i}\int_{-\infty}^{0}\frac{dt}{t(st-p)}\left\{\frac{1}{p^2+\omega_p^2t^2F^{+}(t)}-\frac{1}{p^2+\omega_p^2t^2F^{-}(t)}\right\}\right\}. \tag{36}$$

We may pass to the limit of large time exactly as before and we then easily find the standard result

$$(37)\quad \lim_{t\to\infty}\left(E(x,t)e^{+i\omega t}\right) = \frac{E_o}{1-\frac{\omega_p^2}{\omega^2}} - \frac{E_o}{\pi i}\int_{-\infty}^{0}\frac{dt'\,e^{-ix/t'}}{t'}\left\{\frac{1}{1-\omega_p^2 t'^2 F^-(t'\omega)} - \frac{1}{1-\omega_p^2 t'^2 F^+(t'\omega)}\right\}.$$

IV. Asymptotic Expansions of the Steady State Solution

In this section we consider the asymptotic form of the electric field after a long time

$$(38)\qquad \mathcal{E}(x,t,\omega) \equiv e^{-i\omega t}\lim_{t\to\infty}\left(E(x,t)e^{i\omega t}\right),$$

where the Laplace transform in space of the limit is given by (30). We have obtained this steady state solution by passing to the limit of large time in a time varying solution. However, we may now easily characterize this solution uniquely by the condition that $\mathcal{E}(x,t,\omega)$ possess an analytic continuation into the half-plane $\operatorname{Im}\omega \geq 0$. Such a condition leads to the same Hilbert problem (23) (with $G \equiv 0$, and the factor $(p+i\omega)$ omitted) and ultimately to the same solution (30). Such an analyticity property is typical of causality conditions in scattering theory.

We have already obtained one approximation for $\mathcal{E}(x,t,\omega)$ for large x,

$$(31') \qquad \lim_{x \to \infty} \mathcal{E}(x,t,\omega) = \frac{e^{-i\omega t}}{(1-\omega_p^2/\omega^2)} \left(E_o - \frac{J}{\omega i}\right),$$

where

$$J = 4\pi q \left\{ \int_0^\infty v h^+(v) dv + \int_{-\infty}^0 v h^-(v,t=\infty) dv \right\}$$

but we would like sharper estimates analogous to the results in Section II for the initial value problem. Earlier we were able to derive expansions on the basis of two distinct assumptions. First, we were able to adjust the contour of integration in (3) so that we could take uniformly valid asymptotic expansions of all the functions in question. In order that the error terms be small we required that t be small. In the present case the contour of integration for the inversion of the Laplace transform of $\mathcal{E}(x,t,\omega)$ passes through $s = \infty$, and so the only expansions that we can hope to be uniformly valid are those valid near $s = \infty$. We could indeed find such expansions, which apply for small values of x, by taking a contour of integration far from the origin in the s-plane and expanding the functions. However these expansions are probably none too significant, as the results depend very sensitively on the precise behavior of the distribution functions for small velocity. Further, they do not exhibit any plasma oscillation phenomena and hence we omit them. Our second method of obtaining expansions in section II relied on the observation that the integral (3) was singular at $k\lambda = 0$, and hence there must be approximations possible for $k\lambda$ small. Here the analogous observation is that (26), (30) and (31') are singular at $\omega^2 = \omega_p^2$ and hence we expect interesting results for $1 - \omega_p^2/\omega^2$ small. We treat this limiting case in this section in two parts, for we must examine separately $1 - \omega_p^2/\omega^2$ small and positive and then small and negative. We obtain results relevant for a large enough range of values of x to observe Landau damping of plasma oscillations. As a matter of convenience we restrict ourselves to type I distribution functions.

Our first task is to find appropriate representations for $\Gamma(z,-i\omega)$ for all real ω. We start from (26a), which provides the analytic continuation of $\Gamma(z,-i\omega)$ for all ω, $\omega^2 \neq \omega_p^2$, and we wish to undo the transformation that led to (26a) and return to a form analogous to (24). With $p = -i\omega$, (26a) becomes

$$(26'a)\quad \Gamma(z,-i\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt'}{(t' - z/\omega)} \log\left\{\frac{(1-\omega_p^2 t'^2 F(\omega t'))\omega^2}{\omega_p^2-\omega^2}\right\}$$

$$+ \log\left(\frac{\omega^2-\omega_p^2 z^2 F^{-}(z)}{\omega^2-\omega_p^2}\right),$$

where we have taken Im $z \leq 0$ (and we assume throughout that $\omega > 0$). The reader may readily verify that whenever $\Gamma(z,-i\omega)$ is used in (30) it corresponds to values of z in the lower half-plane or on its boundary, and whenever there is an ambiguity for real ω, the result is the limiting value for ω in the upper half-plane.

We wish to take the section of the integral from minus infinity to zero and deform it through the lower half-plane into the path from zero to plus infinity. Besides the pole in the lower half-plane at $t' = z/\omega$ there may be a branch cut at a zero of the logarithm. For a stable plasma it is readily shown that in the lower (or upper) half-plane $t^2F(t)$ assumes all real values from zero to one exactly once.[5] We denote by $t_\omega^\pm$ the zero of $(\omega^2-\omega_p^2t^2F(t))$ in the upper or lower half-plane, $\omega^2 < \omega_p^2$. To accomplish the transformation described above we use the contour of integration of Fig. 4 and we find

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[5] From the asymptotic expansion it is clear that there is a curve near the imaginary axis at infinity on which $t^2F^\pm(t)$ is real and positive. It is easy to show that the curves enter the origin and there are no other such curves. Hence $t^2F^\pm(t)$ is monotone on the curve and takes all real values from 0 ($t = 0$) to 1 ($t = \infty$).

$$(39) \qquad \Gamma(z,-i\omega) = \frac{1}{2\pi i}\int_0^\infty \left(\frac{dt}{t-z}\right) \log\left\{\frac{\omega^2-\omega_p^2 t^2 F^+(t)}{\omega^2-\omega_p^2 t^2 F^-(t)}\right\} + \begin{cases} \log\left(\frac{z-t_\omega^-}{z}\right) & 0 < \omega < \omega_p \\ 0 & \omega > \omega_p, \end{cases}$$

and in (39) the logarithm tends to zero as t tends to infinity. Hence $\Gamma(z,-i\omega)$ is analytic in z except for a cut on the positive real axis and possibly a branch point in the lower half-plane. We see below that $\Gamma(z,-i\omega)$ is in fact bounded for small z even though there is an apparent singularity at $z = 0$. With this information we may modify the contour of integration in (30) and finally write

$$(40) \qquad \mathcal{E}(x,t) = \frac{e^{-i\omega t}}{2\pi i}\int_{\mathrm{Re}\ s > 0} e^{sx}\, ds I(s),$$

where

$$(41) \quad I(s) = \frac{e^{\Gamma\left(-\frac{i\omega}{s},\, -\, i\omega\right)}}{s\left(1+\frac{\omega_p^2}{s^2}F\left(-\frac{i\omega}{s}\right)\right)}\left\{E_o e^{-\Gamma(0,-i\omega)} + \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{dt'(H^+(\omega t')-H^-(\omega t'))e^{-\Gamma(\omega t',-i\omega)}}{(st'+i)}\right\} .$$

We readily infer that $I(s)$ is analytic in s except for a cut on the positive imaginary axis, a pole at $s = -i\omega/t^+_\omega$ $(0 < \omega < \omega_p)$, and a pole at $s = 0$. We deform the contour according to Fig. 5 to obtain

$$(42) \qquad \mathcal{E}(x,t) = \frac{(E_o - \frac{J}{i\omega})e^{-i\omega t}}{(1 - \omega_p^2/\omega^2)} + \frac{e^{-i\omega t}}{2\pi i}\int_o^{i\infty} e^{sx} ds(I^+(s) - I^-(s))$$

$$+ \begin{cases} \lim\limits_{s \to -i\omega/t^+_\omega} \left\{(s + \frac{i\omega}{t^+_\omega})I(s)\right\} e^{0i\omega(t + x/t^+_\omega)} , & 0 < \omega < \omega_p \\ 0 , & \omega > \omega_p \ . \end{cases}$$

We note that if both $f_o(v)$ and $h^+(v)$ are type I distribution functions then $(I^+(s) - I^-(s)) = \bigcirc(s^n)$ for s small for any n and $I(s) = \bigcirc(\frac{1}{s})$ for s large. Hence for ω fixed and different from ω_p the expansion of $E(x,t)$ for large x is just (31') with an error term $\bigcirc(1/x^n)$ obtained by repeated integration by parts. Again, we believe this to be the only expansion possible for large x which does not depend on the detailed properties of the distribution functions and boundary conditions. In order to obtain expansions for $1 - \omega_p^2/\omega^2$ small we must return to (39) and obtain asymptotic expansions for large z uniform in $(1 - \omega_p^2/\omega^2)$.

We consider first the simpler case of $(1 - \omega_p^2/\omega^2)$ small negative. From the asymptotic expansions of $F^{\pm}(t)$ we see that $\omega^2-\omega_p^2t^2F^{\pm}(t)$ is approximately real and negative for t large and real so that the logarithm in the integrand in (39) is uniformly $\mathcal{O}(1/t^n)$ for any n. Thus the asymptotic expansion gives no difficulties and we find easily

$$(43)\qquad \Gamma(z,-i\omega) = \sum_{n=1}^{N} \frac{\alpha_n}{z^n} + \mathcal{O}(1/z^{N+1}) + \log\left(\frac{z-t_\omega^-}{z}\right)$$

where the coefficients and error estimate are uniformly bounded in $(1 - \omega_p^2/\omega^2)$. The standard methods also show that the integral in (39) is $\log z + \mathcal{O}(1)$ for small z, so that $\Gamma(z,-i\omega)$ is $\mathcal{O}(1)$ for small z. In order to complete our discussion of the expansion of the electric field in this case we need an approximation to $t_\omega^{\pm}$ and we find easily that

$$(44)\qquad t_\omega^{\pm} = \pm i\sqrt{\frac{\beta_2}{1-\frac{\omega^2}{\omega_p^2}}}\left\{1 + \sum_{n=1}^{N} \delta_n^{\pm}\left(1-\frac{\omega^2}{\omega_p^2}\right)^{n/2} + \mathcal{O}\left(1-\frac{\omega^2}{\omega_p^2}\right)^{(N+\frac{1}{2})}\right\}.$$

For this case we almost have the complete answer in (42) since we have an explicit answer plus an integral which we wish to show small. We divide the integral into two integrals, one from 0 to $iR\omega$ and then other from $iR\omega$ to $i\infty$. For the first integral we take the asymptotic expansion to N^{th} order in s (which is identically zero) plus an error term. In the second integral we integrate by parts repeatedly n times and evaluate the end point terms approximately by the asymptotic expansion to Nth order. If we use the expansion (44) explicitly only to lowest order then we find

$$(45) \qquad \mathcal{E}(x,t) = \frac{\left(E_o - \frac{J}{i\omega}\right)}{\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \left(1 - \exp\left(-\,\omega x \sqrt{\frac{\omega_p^2-\omega^2}{\omega_p^2\beta_2}}\right)\right) e^{-i\omega t}$$

$$+\, \mathcal{O}\left(\frac{R^{N+1}\left(1+R(1-\,\omega^2/\omega_p^2)^{-\frac{1}{2}}\right)}{(1-\,\omega^2/\omega_p^2)^2}\right)$$

$$+ \sum_{m=o}^{n} \mathcal{O}\left\{\frac{x^{-n}\,(1-\,\omega^2/\omega_p^2)^{-\frac{1}{2}}}{R^{m+1}((1-\,\omega^2/\omega_p^2)+R^2)^{n-m+1}}\right\} \quad .$$

In Eq. (45) we have by no means the sharpest error estimate possible, but it suffices to show, just as in Section II, that the exponentially damped component of the electric field is in fact observable until it damps significantly and then the non-constant part of the electric field disperses in an irregular manner. We omit the details on the choices of R, N, and n since they are essentially the same as the discussion following in Eq. (14). The exponential damping observable in this case is not "Landau damping" since it is not associated with plasma oscillations and since it would be present in a macroscopic theory as well as in the present kinetic approach.

We now consider the case $1 - \omega_p^2/\omega^2$ small and positive. The expansion of $\Gamma(z,-i\omega)$ for large z is somewhat more complicated. Since $(F^+(t))^* = F^-(t)$ we infer that

$$(39') \qquad \Gamma(z,-i\omega) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t-z} \arg(\omega^2-\omega_p^2 t^2 F^+(t)).$$

From the asymptotic expansion of $F^+(t)$ we infer that $\mathrm{Re}(\omega^2-\omega_p^2 t^2 F^+(t))$ changes from negative to positive as t increases and is large. In fact $\mathrm{Re}(\omega^2-\omega_p^2 t^2 F^+(t))$ changes sign at least one other time since the value is positive at t equals zero, but we are concerned with the changes in sign in the asymptotic region only. We denote by $t_\omega^\pm$ the positive (negative) value of t approximately equal to $\pm(\omega^2/\omega_p^2 - 1)^{\frac{1}{2}}$ for which $\mathrm{Re}(\omega^2-\omega_p^2 t^2 F^+(t)) = 0$. Since $f_o(v)$ is a type I distribution function it follows that in any but an exceedingly narrow interval surrounding t_ω^+

$$\arg(\omega^2-\omega_p^2 t^2 F^+(t)) = \begin{cases} 0 + \mathcal{O}(\frac{1}{t^N}) & t > t_\omega^+ \\ \pi + \mathcal{O}(\frac{1}{t^N}) & t < t_\omega^+ \end{cases} .$$

The width of the interval in which the above estimate fails is less than $\mathcal{O}(1/t^N)$ for any N. Thus if we add and subtract $\pi\eta(t_\omega^+-t)$ from (39') we readily obtain the expansion

$$(48) \qquad \Gamma(z,-i\beta) = \sum_{n=1}^{N} \frac{\alpha_n'}{z^n} + O\left(\frac{1}{z^{N+1}}\right) + \log\left(\frac{z-t_\omega^+}{z}\right)$$

valid for z bounded away from the positive real axis and again the coefficients are uniformly bounded in $(1 - \omega_p^2/\omega^2)$. We note the formal similarity to (43) for the case $\omega < \omega_p$. Again the standard arguments show that $\Gamma(z,-i\omega)$ is $O(1)$ for small z.

We are now prepared to obtain the asymptotic expansion of the electric field. For the values of x which are not too large we use the contour of Fig. 6, where the circle is centered at $s = -i\omega/t_\omega^-$ and is of radius ρ, and

$$\mathcal{E}(x,t) = \frac{\left(\mathcal{E}_0 - \frac{J}{i\omega}\right)e^{-i\omega t}}{(1 - \omega_p^2/\omega^2)} + \frac{e^{-i\omega t}}{2\pi i}\int_C e^{sx}\, ds I(s).$$

We now follow the same procedure as before. We split the integral into two parts, an integral from 0 to $iR\omega$, $R > \rho$, and an integral from $iR\omega$ to infinity. On the first part of the contour we use the asymptotic expansions to N^{th} order terms. On the second part of the contour we integrate by parts n times and use the asymptotic expansions to estimate the error terms. If we carry the explicit answer only to the lowest order non-trivial terms we find

$$(47) \qquad \mathcal{E}(x,t) = \frac{(E_o - J/i\omega)}{(1-\omega_p^2/\omega^2)}\left(1-\exp\left(i\omega x\sqrt{\frac{\omega^2-\omega_p^2}{\beta_2\omega_p^2}}\right)\right)e^{-i\omega t}$$

$$+\,O\left[\frac{R^{N+1}\left(1+R\left(\frac{\omega^2}{\omega_p^2}-1\right)\right)^{-\frac{1}{2}}e^{x\rho}}{\left\{\left(\frac{\omega^2}{\omega_p^2}-1\right)^{\frac{1}{2}}+\frac{\rho}{\omega}\right\}^2\frac{\rho^2}{\omega^2}}\right]$$

$$+\sum_{m=o}^{n} O\left[\frac{x^{-n}\left(\frac{\omega^2}{\omega_p^2}-1\right)^{-\frac{1}{2}}}{R^{m+1}\left\{\left(\frac{\omega^2}{\omega_p^2}-1\right)-R^2\right\}^{n-m+1}}\right]$$

From (47) we easily infer that we may follow the undamped plasma oscillation for a distance of the order of a reciprocal power of $(\omega^2/\omega_p^2 - 1)$. Again we need a more sensitive expansion to obtain any damping. We use the contour of Fig. 6 again except that we move the semi-circle on the right half-plane onto the imaginary axis. We use a Taylor series plus remainder on the sections of the integral from $-i\omega/t_\omega^- - i\rho$ to $-i\omega/t_\omega^- + i\rho$, and integrate by parts n times on the remainder of the contour. We move the section of the contour on which the Taylor series was applied onto the semi-circle in the left half-plane, getting a contribution from the pole near $s = -i\omega/t_\omega^-$, and we integrate by parts n. times. We thus obtain

$$(48) \quad \mathcal{E}(x,t) = \frac{(E_o - J/i\omega)}{(1- \omega_p^2/\omega^2)} \left(1-\exp\left(x\omega\left(i\sqrt{\frac{\omega^2-\omega_p^2}{\beta_2\omega^2}} - \delta\right)\right)\right)e^{-i\omega t}$$

$$+ O\frac{\left(1+\left(\frac{\rho}{\omega}\right)\left(\frac{\omega^2}{\omega_p^2} -1\right)^{-\frac{1}{2}}\right)\left(\frac{\rho}{\omega}\right)^N}{\left[\left(\frac{\omega^2}{\omega_p^2} -1\right)^{\frac{1}{2}} - \frac{\rho}{\omega}\right]\left[i\mathcal{E}+\left(\frac{\omega^2}{\omega_p^2} -1\right)^{\frac{1}{2}}\frac{\rho}{\omega}\right]^2}$$

$$+ \sum_{m=o}^{n} O\left\{\frac{x^{-n}(\omega^2/\omega_p^2 - 1)^{-\frac{1}{2}}}{\left[\left(\frac{\omega^2}{\omega_p^2} -1\right)^{\frac{1}{2}} - \frac{\rho}{\omega}\right]^{m+1}\left[\left(\frac{\omega^2}{\omega_p^2} -1 \right.^{\frac{1}{2}} + \frac{\rho}{\omega}\right]^{n-m+1}\left(\frac{\rho}{\omega}\right)^{n-m+1}}\right\},$$

where

$$(49) \quad \delta = -\frac{\pi}{2}\left(\frac{u^3\partial f_o(u)}{\partial u}\right)_{u = -\left(\frac{\omega^2}{\omega_p^2} -1\right)^{-\frac{1}{2}}} \cong \frac{\mathcal{E}\omega^3}{2\omega_p^2\sqrt{\omega^2/\omega_p^2 -1}} .$$

If in (48) we take $\frac{\rho}{\omega} = \frac{1}{2}(\omega^2/\omega_p^2 -1)^{\frac{1}{2}}$, then the result is essentially the same (16) with $k\lambda$ replaced by $(\omega^2/\omega_p^2 -1)^{\frac{1}{2}}$, and again it is easy to show that it is possible to observe the damping of the plasma oscillations long enough to see their exponential damping, but after the net damping is large there is only a slow dispersion of the electric field to its equilibrium value (31').

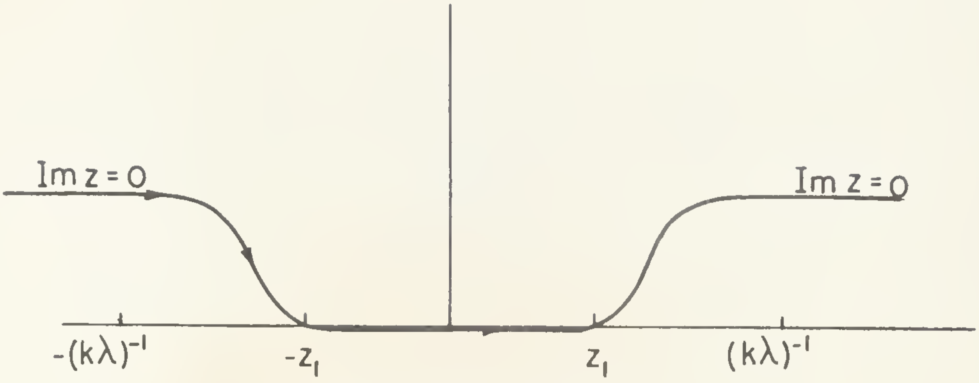

Figure 1 A Contour of Integration

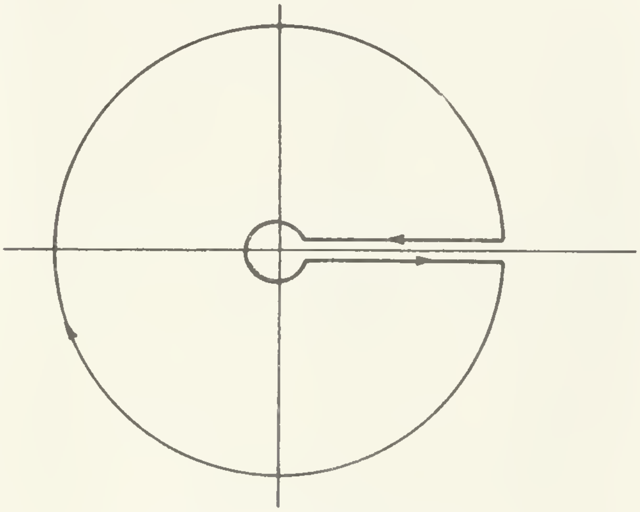

Figure 2 A Contour of Integration

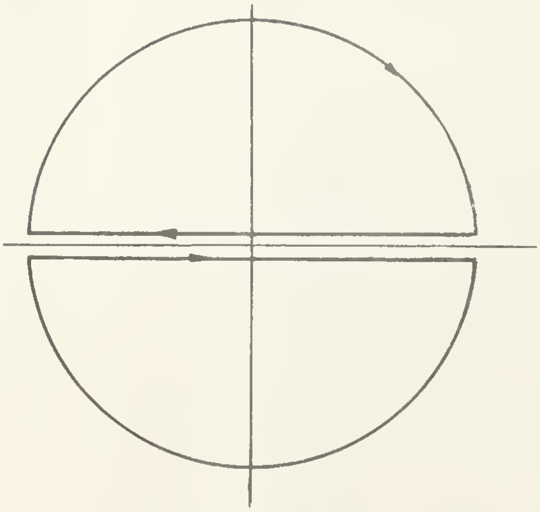

Figure 3 Contours of Integrations

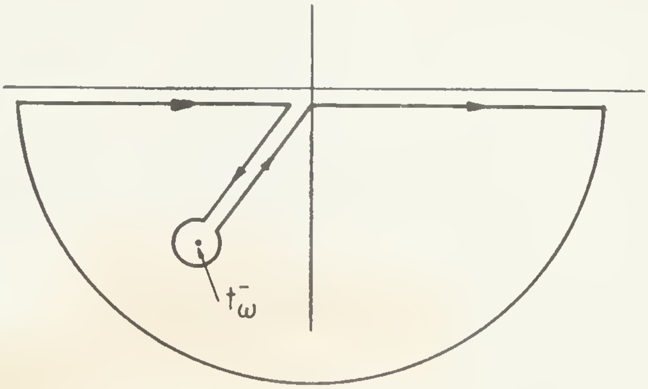

Figure 4 A Contour of Integration

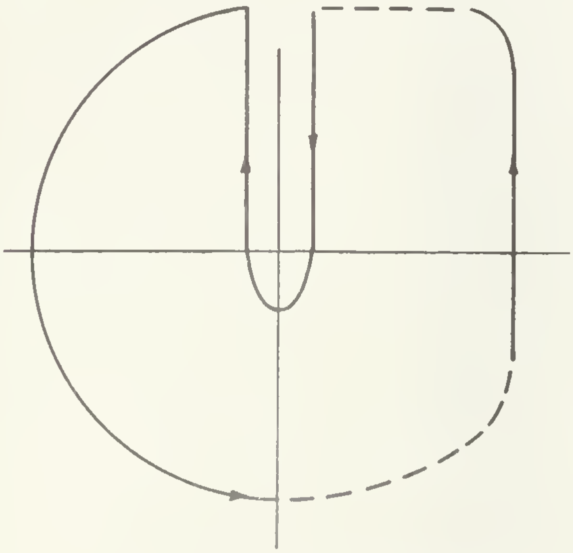

Figure 5 A Contour of Integration

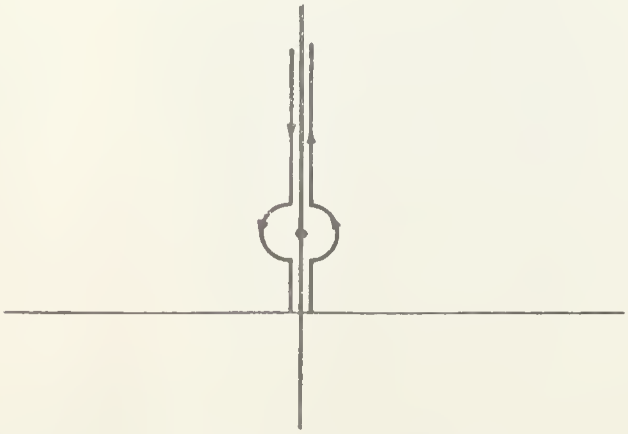

Figure 6 A Contour of Integration

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